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Functions of Class Lip(α, p) and Their Taylor Mean

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The object of this paper is to study the rapidity of convergence of the Taylor mean of the Fourier series of $f(x)$ when $f(x)$ belongs to the class Lip(α, p). We show that it is of Jackson order provided that a suitable integrability condition is imposed upon the function $\varphi_x(t) = \frac{1}{2}\{f(x+t) - 2f(x) + f(x-t)\}$. © 1985 Academic Press, Inc.

1. DEFINITIONS AND NOTATION

Let $f \in L[-\pi, \pi]$ and be periodic with period 2π. Let the Fourier series of f be given by

$$S(x) = \sum_{-\infty}^{\infty} c_m e^{imx}. \tag{1.1}$$

Let the nth partial sum of the series (1.1) be $s_n(x) = \sum_{-n}^{n} c_m \exp(imx)$.

Let $\{a_{nk}\}$ be an infinite matrix defined by

$$\frac{(1-r)^{n+1}\theta^n}{(1-r\theta)^{n+1}} = \sum_{k=0}^{\infty} a_{nk}\,\theta^k \qquad (|r\theta| < 1). \tag{1.2}$$

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The Taylor mean of $\{s_n(x)\}$ is given by

$$T_n^r(f; x) = \sum_{k=0}^{\infty} a_{nk} s_k(x) \tag{1.3}$$

whenever the series on the right of (1.3) is convergent for each $n = 0, 1, \ldots$.

The series (1.1) is said to be Taylor summable to s if

$$\lim_{n \to \infty} T_n^r(f; x) = s.$$

For $p \geqslant 1$, $\delta > 0$, we let $w(g; \delta)$ and $w_p(g; \delta)$ denote the modulus of continuity and integral modulus of continuity, respectively, of an appropriate function g (see [12, pp. 42, 45]).

All norms to be considered in this paper will be L_p $(p \geqslant 1)$ norms. Throughout the paper, norms will be taken *with respect to the variable* x, and the subscript p to L_p norms will generally be omitted. The classes Lip α, Lip(α, p), Lip* α, Lip*(α, p) $(p \geqslant 1)$ will be as usual (see [4, p. 612], also see [12, pp. 42, 45]). The class Lip(α, p) with $p = \infty$ reduces to Lip α.

Throughout the paper we shall let A stand for a positive constant which need not have the same value at each occurrence.

We shall write

$$\varphi_x(t) = \tfrac{1}{2}\{f(x+t) - 2f(x) + f(x-t)\}, \tag{1.4}$$

$$1 - r\exp(2it) = \rho\exp(-2i\theta), \tag{1.5}$$

$$K(n, t) = ((1-r)/\rho)^{n+1} \sin[(n+1)(\theta+t) - (t(n+1)/2)], \tag{1.6}$$

$$a_n = \pi\left\{n + \frac{1}{2} + \frac{n+1}{1-r}r\right\}^{-1}. \tag{1.7}$$

2. Introduction

The Taylor summability transform has been discussed by many authors (see [1–10]). Boehme and Powell [1] have considered generalizations of the Taylor summability transform and the uniform convergence of a linear operator associated with the generalized Taylor transform (see [1, p. 29, Theorem 4.1]). Forbes [3], Ishiguro [7], and Lorch and Newman [9] have considered the Lebesgue constants associated with the Taylor method. Miracle [10] has studied the Gibbs phenomenon for Taylor means. Sufficient conditions for the Taylor summability of the Fourier series (1.1) has been obtained by Holland, Sahney, and Tzimbalario [6].

Hardy and Littlewood [4] have stated without proof that the class of functions Lip(α, p) is identical with the class of functions approximable in the L_p norm with an error $O(n^{-\alpha})$, by trigonometrical polynomials of degree n. With a view to examining the range of values of α and p for which the statement of Hardy and Littlewood holds, Quade [11] has obtained the following, amongst other results:

THEOREM A. *If the function $f(x)$ can be approximated for each $n \geqslant 1$, by a trigonometrical polynomial, $t_n(x)$, of degree n at most, such that $\|f - t_n\| = O(n^{-\alpha})$, $p \geqslant 1$, then*

(i) *if* $0 < \alpha < 1$, $f(x) \in \text{Lip}(\alpha, p)$;

(ii) *if* $\alpha = 1$, $w_p(\delta; f) = O(\delta \log \delta^{-1})$.

Moreover there exist functions for which $\|f - t_n\| = O(n^{-1})$ which do not belong to Lip$(1, p)$.

THEOREM B. *If $f(x) \in \text{Lip}(\alpha, p)$, $p \geqslant 1$, $0 < \alpha \leqslant 1$, then, for any integer n, $f(x)$ may be approximated in L_p by a trigonometric polynomial, $t_n(x)$, of order n such that*

$$\|f - t_n\| = O(n^{-\alpha}).$$

With a view to obtaining the degree of approximation of the Taylor mean $T_n^r(f; x)$ to $f \in \text{Lip}\, \alpha$ $(0 < \alpha < 1)$, Chui and Holland [2] have proved:

THEOREM C. *If $f(x) \in \text{Lip}\, \alpha$, $0 < \alpha < 1$, and*

$$\int_{a_n}^{(a_n)^\beta} \frac{|\varphi_x(t) - \varphi_x(t + a_n)|}{t} e^{-nrt^2/2(1-r)^2} \, dt = O(n^{-\alpha}) \tag{2.1}$$

uniformly in x, where $(1 + \alpha)/(3 + \alpha) \leqslant \beta < \frac{1}{2}$ and a_n is given by (1.7), *then*

$$\max_{0 \leqslant x \leqslant 2\pi} |T_n^r(f; x) - f(x)| = O(n^{-\alpha}). \tag{2.2}$$

They have remarked that since the Lebesgue constants for the Taylor method diverge as $n \to \infty$ in order to get the degree of convergence of order $n^{-\alpha}$, $f \in \text{Lip}\, \alpha$ is not adequate.

The object of this paper is to obtain the degree of convergence of $T_n^r(f; x)$ to $f(x)$ in the L_p norm when $f \in \text{Lip}(\alpha, p)$ (Theorem 1). Since we find that the error is of order $n^{-\gamma}$ $(0 < \gamma < \frac{1}{2})$, we have obtained a subclass of Lip(α, p) which satisfies an integrability condition analogous to (2.1), and for which $\|T_n^r(f; x) - f(x)\| = O(n^{-\alpha})$ $(0 < \alpha < 1)$ (see Theorem 4).

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We shall need the following lemmas:

LEMMA 1 (see [3]). *For $0<r<1$, $|r\theta|<1$, ρ given by* (1.5), *we have*

$$((1-r)/\rho)^n \leqslant \exp(-Ant^2) \qquad (0 \leqslant t \leqslant \pi), \tag{3.1}$$

and

$$\{((1-r)/\rho)^n - \exp(-nr^2/2(1-r)^2)\} = O(nt^4) \qquad (t>0). \tag{3.2}$$

LEMMA 2 (see [10]). *Let r, θ be as in Lemma* 1. *Then*

$$\{\theta - rt/(1-r)\} \leqslant At^3 \qquad (0 \leqslant t \leqslant \pi/2). \tag{3.3}$$

LEMMA 3 [5, p. 148, 6.13.9]. *If $h(x,t)$ is a function of two variables defined for $0 \leqslant t \leqslant \pi$, $0 \leqslant x \leqslant 2\pi$, then*

$$\left\| \int h(x,t)\,dt \right\|_p \leqslant \int \|h(x,t)\|_p\,dt \qquad (p>1).$$

LEMMA 4 [4, Theorem 5(ii), p. 627]. *Suppose that $f \in \mathrm{Lip}(\alpha, p)$ where $p \geqslant 1$, $0<\alpha\leqslant 1$, $\alpha p>1$. Then f is equal to a function $g \in \mathrm{Lip}(\alpha - 1/p)$ almost everywhere.*

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Our first theorem gives an L_p estimate for the error in approximating an $f \in \mathrm{Lip}(\alpha, p)$ by $T_n^r(f)$.

THEOREM 1. *If $f \in \mathrm{Lip}(\alpha, p)$, $0<\alpha\leqslant 1$, $p>1$, then*

$$\|T_n^r(f;x) - f(x)\| = O(n^{-\alpha\beta}) \tag{4.1}$$

for $0<\beta<\frac{1}{2}$.

This theorem will be deduced from the following general theorem:

THEOREM 2. *If $f \in L_p$ $(p>1)$, then, for $0<\beta<1$,*

$$\|T_n^r(f;x) - f(x)\| = O\left(w_p\left(\frac{1}{n};f\right)\right) + O\left(\int_{a_n}^{(a_n)^\beta} \frac{w_p(t;f)}{t}\,dt\right) + O(n^\beta \exp(-An^{1-2\beta})). \tag{4.2}$$

Since $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leqslant 1$, $p > 1$, implies that $w_p(\delta; f) = O(\delta^{\alpha})$ $(\delta > 0)$, and since for $0 < \beta < \frac{1}{2}$

$$n^{\beta} \exp(-An^{1-2\beta}) = O(n^{-\alpha}), \tag{4.3}$$

to deduce Theorem 1 from Theorem 2, it is enough to show that

$$\int_{a_n}^{(a_n)^{\beta}} \frac{w_p(t)\, dt}{t} = O(n^{-\alpha\beta}). \tag{4.4}$$

Equation (4.4) follows from the fact that

$$\int_{a_n}^{(a_n)^{\beta}} t^{\alpha-1}\, dt = O(n^{-\alpha\beta}) + O(n^{-\alpha}).$$

Proof of Theorem 2. Since

$$s_k(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{\{f(x+t) + f(x-t)\}}{\sin(t/2)} \sin((2k+1)\, t/2)\, dt,$$

and since

$$\sum_{k=0}^{n} a_{nk} \sin(2k+1)\, u = \left(\frac{1-r}{\rho}\right)^{n+1} \sin\left[(n+1)\left\{2(u+\theta) - \frac{u}{n+1}\right\}\right],$$

we have

$$\begin{aligned} T_n^r(f; x) - f(x) &= \frac{1}{\pi}\left\{\int_0^{a_n} + \int_{a_n}^{(a_n)^{\beta}} + \int_{(a_n)^{\beta}}^{\pi}\right\} \frac{\varphi_x(t)\, K(n, t)\, dt}{\sin(t/2)} \\ &= I_1 + I_2 + I_3, \qquad \text{say.} \end{aligned} \tag{4.5}$$

By the Minkowski inequality

$$\| T_n^r(f; x) - f(x)\| \leqslant \|I_1\| + \|I_2\| + \|I_3\|. \tag{4.6}$$

Now, since $|1-r| \leqslant \rho$, $\sin t/2 \geqslant t/\pi$ when $0 < t \leqslant \pi$, we have, by Lemma 3, that

$$\|I_1\| \leqslant \int_0^{a_n} \frac{\|\varphi_x(t)\|}{t} \left\{\sin\left(n + \frac{1}{2}\right) t + (n+1)\, \theta\right\} dt. \tag{4.7}$$

By Lemma 2 and the fact that sin $nt \leqslant nt$ $(0 \leqslant t \leqslant a_n)$, the integral on the right of (4.7) does not exceed

$$\int_0^{a_n} \frac{w_p(t;f)}{t} \left\{ \left(n+\frac{1}{2}\right) t + (n+1)\left(At^3 + \frac{rt}{1-r}\right)\right\} dt$$
$$= O\left(n \int_0^{a_n} w_p(t;f)\, dt\right)$$

since $t^3 \leqslant t$ $(0 \leqslant t \leqslant a_n)$. Hence

$$\|I_1\| = O(w_p(n^{-1};f)). \tag{4.8}$$

Also, by Lemma 1 ((3.1))

$$K(n, t) = O(\exp(-Ant^2)), \tag{4.9}$$

we have by Lemma 3 and (4.9)

$$\|I_3\| = O\left(\int_{b_n}^{\pi} \frac{\omega_p(t;f)}{t} e^{-Ant^2}\, dt\right)$$
$$= O(n^{\beta} \exp(-An^{1-2\beta})), \tag{4.10}$$

since $w_p(t;f) \leqslant w_p(\pi;f) = O(1)$.

Finally, by Lemma 3 and the fact that $|\sin x| \leqslant 1$ for all x, we have

$$\|I_2\| = O\left(\int_{a_n}^{(a_n)^{\beta}} \frac{\|\varphi_x(t)\|}{t}\, dt\right)$$
$$= O\left(\int_{a_n}^{(a_n)^{\beta}} \frac{w_p(t;f)}{t}\, dt\right). \tag{4.11}$$

On collecting the estimates from (4.8), (4.10), and (4.11) we get (4.2).

COROLLARY 1. *If* $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \leqslant 1$, *then*

$$T_n^r(f; x) - f(x) = O(n^{-\beta\alpha}) \qquad (0 < \beta\alpha < \tfrac{1}{2}) \tag{4.12}$$

uniformly in x *almost everywhere.*

The corollary follows from Theorem 1 by taking $p = \infty$ in (4.1) and the fact that $\mathrm{Lip}(a, p) = \mathrm{Lip}\,\alpha$ when $p = \infty$.

Since $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$ implies $f \in \mathrm{Lip}(\alpha, p)$ $(0 < \alpha \leqslant 1,\ p > 1)$, it is interesting to estimate the expression on the left of (4.12) when $f \in \mathrm{Lip}(\alpha, p)$. We have the following result in that direction:

THEOREM 3. *If* $f \in \text{Lip}(\alpha, p)$, $0 < \alpha \leqslant 1$, $p > 1$, $\alpha p > 1$,

$$T_n^r(f; x) - f(x) = O(n^{-(\alpha - 1/p)\beta}) \qquad (0 < \beta < \tfrac{1}{2}) \tag{4.13}$$

uniformly in x *almost everywhere.*

Proof of Theorem 3. In the notations of Theorem 2,

$$T_n^r(f; x) - f(x) = I_1 + I_2 + I_3, \tag{4.14}$$

as in (4.5).

By Lemma 4, the hypothesis $f \in \text{Lip}(\alpha, p)$ implies that there exists a function $g \in \text{Lip}(\alpha - 1/p)$ such that $f = g$ almost everywhere. Hence, we can conclude that almost everywhere

$$\varphi_x(t) = O(t^{\alpha - 1/p}). \tag{4.15}$$

Using arguments similar to those used in estimating I_1 (without using Lemma 3),

$$\begin{aligned} I_1 &= O\left(\int_0^{a_n} \frac{|\varphi_x(t)|}{t}\, nt\, dt\right) \\ &= O\left(n \int_0^{a_n} t^{\alpha - 1/p}\, dt\right) \\ &= O(n^{-\alpha + 1/p}) \end{aligned}$$

almost everywhere, by (4.15) and (1.7).

In a similar manner one can modify the estimates of I_2 and I_3 and obtain

$$\begin{aligned} I_2 &= O(n^{-(\alpha - 1/p)\beta}) + O(n^{-\alpha + 1/p}) \\ &= O(n^{-(\alpha - 1/p)\beta}) \end{aligned} \tag{4.16}$$

and

$$I_3 = O(n^{\beta} \exp(-An^{1-2\beta})) = O(n^{-(\alpha - 1/p)\beta}). \tag{4.17}$$

The theorem follows from (4.15), (4.16), and (4.17).

Remark. Theorem 3 is a result of the type considered by Izumi [8].

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In this section we shall determine a subclass of the class of functions in Lip(α, p) for which the error in approximating a function by the Taylor mean of its Fourier series is of Jackson order.

Precisely, we shall prove

THEOREM 4. *If* $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha < 1$, $p > 1$, *and*

$$\tilde{I} := \int_{a_n}^{(a_n)^\beta} \frac{\|\varphi_x(t) - \varphi_x(t + a_n)\|}{t} e^{-nrt^2/2(1-r)^2}\, dt = O(n^{-\alpha}) \tag{5.1}$$

where $(1+\alpha)/(3+\alpha) \leqslant \beta < \frac{1}{2}$ *and* a_n *is as in* (1.7), *then*

$$\|T_n^r(f; x) - f(x)\| = O(n^{-\alpha}).$$

THEOREM 5. *Let* $f \in L_p$ $(p > 1)$ *and satisfy the conditions*

$$w_p(t; f)/t^\delta \text{ is a decreasing function of } t \text{ in } (0 \leqslant t \leqslant \pi) \text{ for } 0 < \delta < 1, \tag{5.2}$$

and

$$\tilde{I} = O(w_p(n^{-1}; f)), \tag{5.3}$$

where $\tilde{I}$ *is as in* (5.1) *and* $(1+\delta)/(3+\delta) \leqslant \beta < \frac{1}{2}$. *Then*

$$\|T_n^r(f; x) - f(x)\| = O(w_p(n^{-1}; f)) + O(n^\beta \exp(-An^{1-2\beta})). \tag{5.4}$$

In order to avoid repetitions we first prove Theorem 5 and then deduce Theorem 4 from it by appropriate reasoning.

Proof of Theorem 5. As in Theorem 2 we have

$$\|T_n^r(f; x) - f(x)\| \leqslant \|I_1\| + \|I_2\| + \|I_3\|, \tag{5.5}$$

where I_1, I_2, I_3 are as in (4.5).

In view of the estimates obtained in (4.8) and (4.10), for I_1 and I_3, respectively, it is enough to show that

$$\|I_2\| = O(w_p(n^{-1}; f)). \tag{5.6}$$

Let us write $b_n = (a_n)^\beta$ and I_2 as

$$I_2 = I_{2,1} + I_{2,2}$$

where

$$I_{2,1} = \frac{2}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t)}{t} e^{-nrt^2/2(1-r)^2} \sin\left\{\left(n + \frac{1}{2}\right) t + (n+1)\theta\right\} dt$$

and

$$I_{2,2}=\frac{1}{\pi}\int_{a_n}^{b_n}\varphi_x(t)\left[\frac{1}{\sin t/2}\left(\frac{1-r}{\rho}\right)^{n+1}-\frac{1}{t/2}e^{-nrt^2/2(1-r)^2}\right]$$
$$\times\sin\left\{\left(n+\frac{1}{2}\right)t+(n+1)\theta\right\}dt.$$

By Minkowski's inequality

$$\|I_{2,2}\|\leqslant\|I_{2,21}\|+\|I_{2,22}\|, \tag{5.7}$$

where

$$\|I_{2,21}\|=\int_{a_n}^{b_n}\|\varphi_x(t)\|\,\frac{2}{t}\left|\left\{\left(\frac{1-r}{\rho}\right)^{n+1}-e^{-nrt^2/2(1-r)^2}\right\}\right|dt \tag{5.8}$$

and

$$\|I_{2,22}\|=\int_{a_n}^{b_n}\|\varphi_x(t)\|\left|\frac{2}{t}-\operatorname{cosec}(t/2)\right|\left(\frac{1-r}{\rho}\right)^{n+1}dt. \tag{5.9}$$

By Lemma 1 ((3.2)),

$$\|I_{2,21}\|=O\left(\int_{a_n}^{b_n}\frac{w_p(t;f)}{t}(n+1)\,t^4\,dt\right). \tag{5.10}$$

Since $w_p(t;f)/t^\delta$ is non-increasing for $0<\delta\leqslant 1$, we have

$$\begin{aligned}\|I_{2,21}\|&=O\{(n+1)\,a_n^{-\delta}b_n^{4+\delta}w_p(a_n;f)\}\\&=O(w_p(1/n;f))\end{aligned}$$

since $(n+1)^{-\{\beta(4+\delta)-(1+\delta)\}}=O(1)$ for $\beta\geqslant(1+\delta)/(4+\delta)$, a condition which is satisfied.

Since

$$2/t-\operatorname{cosec} t/2=O(t)$$

and $(1-r)/\rho\leqslant 1$, we have

$$\begin{aligned}\|I_{2,22}\|&=O\left(\int_{a_n}^{b_n}tw_p(t;f)\right)=O(w_p(a_n;f)\,n^\delta/n^{(2+\delta)\beta})\\&=O\left(w_p\left(\frac{1}{n};f\right)\right),\end{aligned} \tag{5.11}$$

by using the facts that $w_p(t; f)/t^\delta$ is non-increasing for $0<\delta<1$ and that $\beta \geqslant \delta/(2+\delta)$.

Finally,

$$I_{2,1} = I_{2,11} + I_{2,12} + I_{2,13}, \tag{5.12}$$

where

$$I_{2,11} = \frac{1}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t) - \varphi_x(t+a_n)}{t} e^{-nrt^2/2(1-r)^2} \sin(a_n^{-1}\pi t)\, dt,$$

$$I_{2,13} = \frac{1}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t)}{t} e^{-nrt^2/2(1-r)^2}$$
$$\times \left[\sin\left\{\left(n+\frac{1}{2}\right) t + (n+1)\theta\right\} - \sin\left\{n + \frac{1}{2} + \frac{n+1}{1-r} r\right\} t \right] dt,$$

$$I_{2,12} = \frac{1}{\pi} \int_{a_n}^{b_n} \frac{\varphi_x(t+a_n)}{t} \left[e^{-nrt^2/2(1-r)^2} - e^{-nr(t+a_n)^2/2(1-r)^2}\right]$$
$$\times \sin(a_n^{-1}\pi t)\, dt.$$

By Lemma 3,

$$\|I_{2,11}\| = O\left(\int_{a_n}^{b_n} \frac{\|\varphi_x(t) - \varphi_x(t+a_n)\|}{t} e^{-nrt^2/2(1-r)^2}\, dt\right)$$
$$= O\left(w_p\left(\frac{1}{n}; f\right)\right),$$

by hypothesis.

By the mean value theorem

$$e^{-nrt^2/2(1-r)^2} - e^{-nr(t+a_n)^2/2(1-r)^2}$$
$$= na_n r\xi \exp(-nr\xi^2/2(1-r)^2)/(1-r)^2, \tag{5.13}$$

for some ξ such that $t<\xi<t+a_n<2t$. Hence the expression in the left of (5.13) is $O(t)$. Substituting this in $\|I_{2,12}\|$ after using Lemma 3 and the fact that $w_p(t; f)/t^\delta$ $(0<\delta<1)$ does not increase, we have

$$\|I_{2,12}\| = O\left(\int_{a_n}^{b_n} \|\varphi_x(t+a_n)\|\, dt\right)$$
$$= O\left(\int_{a_n}^{b_n} w_p(t+a_n; f)\, dt\right) = O\left(\frac{w_p(2a_n; f)}{a_n^\delta} \int_0^{b_n} t^\delta\, dt\right)$$
$$= O\left(w_p\left(\frac{1}{n}; f\right)\right) O\left(\frac{1}{n^{\beta(1+\delta)-\delta}}\right) = O\left(w_p\left(\frac{1}{n}; f\right)\right), \tag{5.14}$$

since $\beta \geqslant \delta/(1+\delta)$.

Further, for $a_n \leqslant t \leqslant b_n$

$$\sin\left\{\left(n+\frac{1}{2}\right)t+(n+1)\theta\right\}-\sin\left\{n+\frac{1}{2}+\frac{n+1}{1-r}r\right\}t=O\left((n+1)\left|\theta-\frac{rt}{1-r}\right|\right)$$

we get as before

$$\|I_{2,13}\| = O\left((n+1)\int_{a_n}^{b_n}\frac{w_p(t;f)}{t}\left|\theta-\frac{rt}{1-r}\right|dt\right)$$

$$= O\left((n+1)\int_{a_n}^{b_n} w_p(t;f)\,t^2\,dt\right), \tag{5.15}$$

by Lemma 2.

Since $w_p(t:f)/t^{\delta}$ $(0<\delta<1)$ is non-increasing, the expression on the right of (5.15) is

$$O\left((n+1)\,a_n^{-\delta} w_p(a_n;f)\int_0^{b_n} t^{2+\delta}\,dt\right\}$$

$$= O(w_p(n^{-1};f)) \tag{5.16}$$

since $\beta \geqslant (1+\delta)/(3+\delta)$.

Thus on collecting the estimates we get (5.6).

This completes the proof.

Proof of Theorem 4. Since $0<\alpha<1$, choose δ such that $0<\alpha<\delta<1$. Since $\varphi_x(t)=O(t^{\alpha})$ when $f\in \text{Lip}(\alpha, p)$, we use t^{α} in place of $w_p(t;f)$ in the proof of Theorem 4. Now that, without choice of δ, $t^{\alpha-\delta} \ddagger$ we can modify the proof of Theorem 5 to get Theorem 4 after noting $f\in \text{Lip}(\alpha, p)$ means $w_p(\eta, f)=O(\eta^{\alpha})$ $(\eta>0)$ and

$$n^{\beta}\exp(-An^{1-2\beta})=O(n^{\alpha}), \qquad (\beta<\tfrac{1}{2}).$$

Our next result is an analogue of Theorem 3.

THEOREM 6. *If* $f\in \text{Lip}(\alpha, p)$, $0<\alpha<1$, $p>1$, $\alpha p>1$, *and*

$$\int_{a_n}^{(a_n)^{\beta}}\frac{|\varphi_x(t)-\varphi_x(t+a_n)|}{t}e^{-nrt^2/2(1-r)^2}\,dt=O(n^{-\alpha+1/p}), \tag{5.17}$$

uniformly in x *where* $(1+\alpha)/(3+\alpha)\leqslant\beta<\frac{1}{2}$ *and* a_n *is as in* (1.7), *then*

$$T_n^r(f;x)-f(x)=O(n^{-\alpha+1/p}), \tag{5.18}$$

uniformly in x *almost everywhere.*

Proof of Theorem 6 is similar to that of Theorem 3 and hence we omit it.

COROLLARY 2. *If $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ and if the integral on the left of (5.17) is of order $n^{-\alpha}$ uniformly in x then*

$$T_n^r(f;x)-f(x)=O(n^{-\alpha})$$

uniformly in x almost everywhere.

The corollary follows from Theorem 6 by making $p \to \infty$.

6. REMARKS

1. The results obtained in this paper hold when "O" is replaced by "o" and the classes $\mathrm{Lip}(\alpha, p)$ and $\mathrm{Lip}\,\alpha$ are replaced by $\mathrm{Lip}^*(\alpha, p)$ and $\mathrm{Lip}^*\,\alpha$, respectively.

2. We have not been able to characterize the class of functions for which

$$\|T_n^r(f;x)-f(x)\|=O(n^{-\alpha}).$$

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